Useful summation formulas and rules

$\Sigma_{l\le i\le u} 1 = 1+1+\cdots+1 = u - l + 1$

In particular, $\Sigma_{1\le i\le u} 1 = n - 1 + 1 = n \in \Theta(n)$

$\Sigma_{1\le i\le n}\, i = 1+2+\cdots+n = n(n+1)/2 \approx n^2/2 \in \Theta(n^2)$

$\Sigma_{1\le i\le n}\, i^2 = 1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6 \approx n^3/3 \in \Theta(n^3)$

$\Sigma_{0\le i\le n}\, a^i = 1 + a + \cdots + a^n = (a^{n+1} - 1)/(a - 1)$ for any $a \ne 1$

In particular, $\Sigma_{0\le i\le n}\, 2^i = 2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1 \in \Theta(2^n)$

$\Sigma(a_i \pm b_i) = \Sigma a_i \pm \Sigma b_i$ $\quad \Sigma c a_i = c\Sigma a_i$ $\quad \Sigma_{l\le i\le u} a_i = \Sigma_{l\le i\le m} a_i + \Sigma_{m+1\le i\le u} a_i$

Cyclic Hanoi

- **Recursively move (n-1) FROM->TO**
- **Move n FROM->WITH**
  - **Recursively move(n-2) TO->WITH**
  - **Move n-1 TO->FROM**
  - **Recursively move(n-2) WITH->FROM**
- **Move n WITH->TO**
- **Recursively move (n-1) FROM->TO**

Cyclic Hanoi – in a single funtion

- If(n<1) do nothing. Just return.
- Recursively move (n-1) FROM->TO
- Move n FROM->WITH
- If(n>1)
  - Recursively move(n-2) TO->WITH
  - Move n-1 TO->FROM
  - Recursively move(n-2) WITH->FROM
- Move n WITH->TO
- Recursively move (n-1) FROM->TO

Cyclic Hanoi

How many times does this function get called?

- Recursively move (n-1) FROM->TO → M(n-1)
- Move n FROM->WITH
  - Recursively move(n-2) TO->WITH → M(n-2)
  - Move n-1 TO->FROM
  - Recursively move(n-2) WITH-FROM → M(n-2)
- Move n WITH->TO
- Recursively move (n-1) FROM->TO → M(n-1)

$$M(n) = 2M(n-1) + 2M(n-2) + 1$$

Non-homogeneous!

Recall: Solving the recurrence

- **In general solution to the inhomogeneous problem is equal to**
  - **the sum of solution to homogenous problem**
  - **plus solution only to the inhomogeneous part.**
- **The undetermined coefficients of the solution for the homogenous problem are used to satisfy the initial conditions.**

In this case, $A(n) = B(n) + I(n)$ where

$A(n)$ is solution to complete inhomogeneous problem

$B(n)$ is solution to homogeneous problem

$I(n)$ solution to only the inhomogeneous part of the problem

Start with master theorem for homogeneous part

- $M(n) = 2M(n-1) + 2M(n-2)$ – Homogeneous part
  - Roots: $(1+\sqrt{3})$ and $(1-\sqrt{3})$
  - $M(n) = (1/(2*\sqrt{3}))\,[(1+\sqrt{3})^n - (1-\sqrt{3})^n]$ -- old formula in excel
- $M(n) = (1/(2*\sqrt{3}))\,[(1+\sqrt{3})^n - (1-\sqrt{3})^n] + I(n)$
  - $M(n) = (1/(2*\sqrt{3}))\,[(1+\sqrt{3})^n - (1-\sqrt{3})^n] + M(n-1)$
- Use backwards substitution:
  - $M(n) = M(n-1) + (1/(2*\sqrt{3}))\,[(1+\sqrt{3})^n - (1-\sqrt{3})^n]$
  - $M(n) = M(n-n) + (1/(2*\sqrt{3}))$

    $[\ (1+\sqrt{3})^n + (1+\sqrt{3})^{n-1} + \ldots + (1+\sqrt{3})^0$ -> $((1+\sqrt{3})^{n+1} - 1)/\sqrt{3}$

    $-\ [(1-\sqrt{3})^n + (1-\sqrt{3})^{n-1} + \ldots + (1-\sqrt{3})^0]$ -> $((1-\sqrt{3})^{n+1} - 1)/\sqrt{3}$ (Note: positive)

    $]$
  - Remember: $1 + a + \cdots + a^n = (a^{n+1} - 1)/(a - 1)$ for any $a \neq 1$
  - $M(n) = (1/6)\,[(1+\sqrt{3})^{n+1} + (1-\sqrt{3})^{n+1} - 2]$ -- new formula in excel