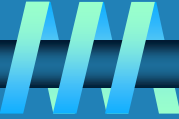


Useful summation formulas and rules



$$\sum_{l \leq i \leq u} 1 = 1 + 1 + \cdots + 1 = u - l + 1$$

In particular, $\sum_{1 \leq i \leq n} 1 = n - 1 + 1 = n \in \Theta(n)$

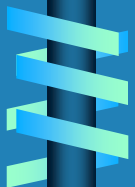
$$\sum_{1 \leq i \leq n} i = 1 + 2 + \cdots + n = n(n+1)/2 \approx n^2/2 \in \Theta(n^2)$$

$$\sum_{1 \leq i \leq n} i^2 = 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6 \approx n^3/3 \in \Theta(n^3)$$

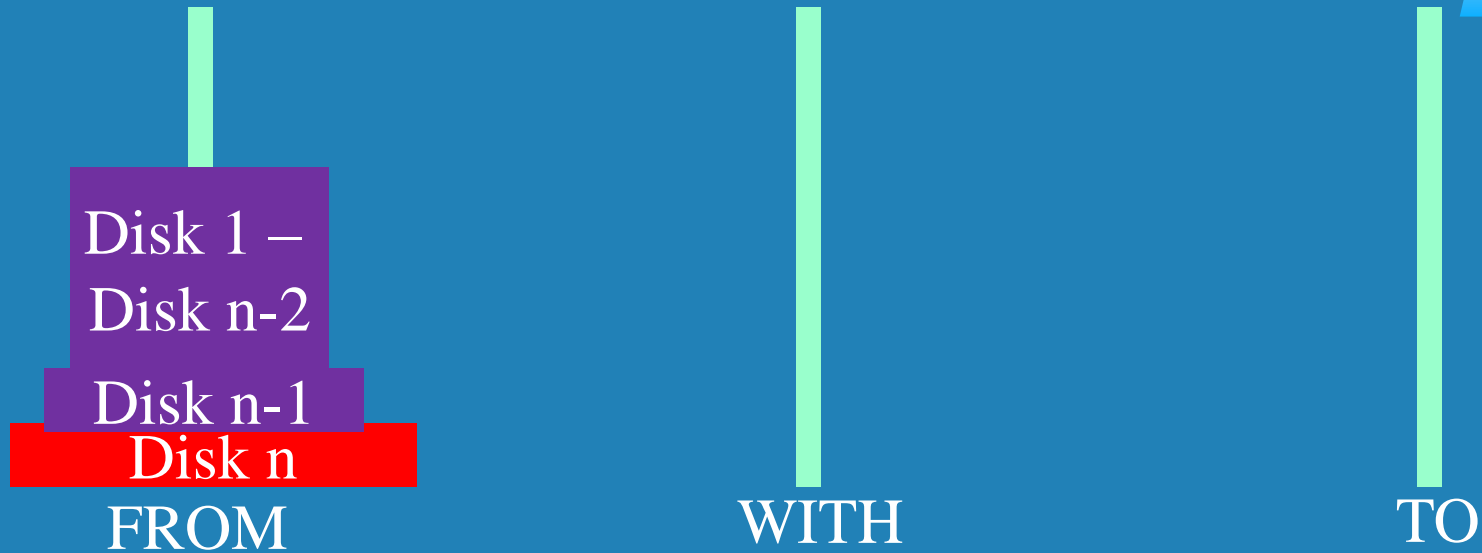
$$\sum_{0 \leq i \leq n} a^i = 1 + a + \cdots + a^n = (a^{n+1} - 1)/(a - 1) \text{ for any } a \neq 1$$

In particular, $\sum_{0 \leq i \leq n} 2^i = 2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1 \in \Theta(2^n)$

$$\sum (a_i \pm b_i) = \sum a_i \pm \sum b_i \quad \sum c a_i = c \sum a_i \quad \sum_{l \leq i \leq u} a_i = \sum_{l \leq i \leq m} a_i + \sum_{m+1 \leq i \leq u} a_i$$

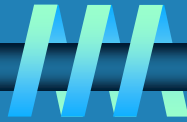


Cyclic Hanoi



- ∞ **Recursively move (n-1) FROM->TO**
- ∞ **Move n FROM->WITH**
 - **Recursively move(n-2) TO->WITH**
 - **Move n-1 TO->FROM**
 - **Recursively move(n-2) WITH->FROM**
- ∞ **Move n WITH->TO**
- ∞ **Recursively move (n-1) FROM->TO**

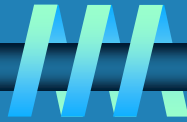
Cyclic Hanoi – in a single function



- ⌚ **If($n < 1$) do nothing. Just return.**
- ⌚ **Recursively move ($n-1$) FROM- \rightarrow TO**
- ⌚ **Move n FROM- \rightarrow WITH**
- ⌚ **If($n > 1$)**
 - **Recursively move($n-2$) TO- \rightarrow WITH**
 - **Move $n-1$ TO- \rightarrow FROM**
 - **Recursively move($n-2$) WITH- \rightarrow FROM**
- ⌚ **Move n WITH- \rightarrow TO**
- ⌚ **Recursively move ($n-1$) FROM- \rightarrow TO**



Cyclic Hanoi



How many times does this function get called?

↻ Recursively move (n-1) FROM->TO → $M(n-1)$

↻ Move n FROM->WITH

- Recursively move(n-2) TO->WITH → $M(n-2)$
- Move n-1 TO->FROM
- Recursively move(n-2) WITH-FROM → $M(n-2)$

↻ Move n WITH->TO

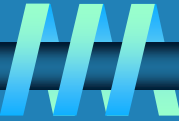
↻ Recursively move (n-1) FROM->TO → $M(n-1)$

$$M(n) = 2M(n-1) + 2M(n-2) + 1$$

Non-homogeneous!



Recall: Solving the recurrence



- ∞ In general solution to the inhomogeneous problem is equal to
 - the sum of solution to **homogenous** problem
 - plus solution only to the inhomogeneous part.
- ∞ The undetermined coefficients of the solution for the homogenous problem are used to satisfy the initial conditions.

In this case, $A(n) = B(n) + I(n)$ where

$A(n)$ is solution to complete inhomogeneous problem

$B(n)$ is solution to homogeneous problem

$I(n)$ solution to only the inhomogeneous part of the problem



Start with master theorem for homogeneous part



∞ $M(n) = 2M(n-1) + 2M(n-2)$ – Homogeneous part

- Roots: $(1+\sqrt{3})$ and $(1-\sqrt{3})$
- $M(n) = (1/(2*\sqrt{3})) [(1+\sqrt{3})^n - (1-\sqrt{3})^n]$ -- old formula in excel

∞ $M(n) = (1/(2*\sqrt{3})) [(1+\sqrt{3})^n - (1-\sqrt{3})^n] + I(n)$

- $M(n) = (1/(2*\sqrt{3})) [(1+\sqrt{3})^n - (1-\sqrt{3})^n] + M(n-1)$

∞ Use backwards substitution:

- $M(n) = M(n-1) + (1/(2*\sqrt{3})) [(1+\sqrt{3})^n - (1-\sqrt{3})^n]$
- $M(n) = M(n-n) + (1/(2*\sqrt{3}))$
 $[(1+\sqrt{3})^n + (1+\sqrt{3})^{n-1} + \dots + (1+\sqrt{3})^0$ $\rightarrow ((1+\sqrt{3})^{n+1} - 1)/\sqrt{3}$
 $- [(1-\sqrt{3})^n + (1-\sqrt{3})^{n-1} + \dots + (1-\sqrt{3})^0]$ $\rightarrow ((1-\sqrt{3})^{n+1} - 1)/\sqrt{3}$ (Note: positive)
 $]$
- **Remember:** $1 + a + \dots + a^n = (a^{n+1} - 1)/(a - 1)$ for any $a \neq 1$
- $M(n) = (1/6) [(1+\sqrt{3})^{n+1} + (1-\sqrt{3})^{n+1} - 2]$ -- new formula in excel

